Information Arrival and Real Investment

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First version:September 1993

Current version: March, 1995

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†I would like to thank Bent Christensen, Robert Cumby and Stephen Figlewski. I acknowledge comments from Sanjiv Das, Apoo Koticha, Massoud Mussavian, N.R.Prabhala and participants at the 1993 European Finance Association conference. Of course, all errors are mine. Please address comments and suggestions to the author at the School of Management, University of Massachusetts, Amherst, MA 01002. Phone:413-545-5643. Email: nkapadia@som.umass.edu.

Abstract

Information Arrival and Real Investment

We analyze the investment decision of the firm under uncertainty. The objective of the paper is to create a generic framework to investigate the respective roles of irreversible or sunk costs of investment and the nature of the resolution of uncertainty in determining the behavior of the hurdle rate. We show analytically that the mode of information arrival is the sole factor that determines the behavior of the option-adjusted hurdle rate. Thus, if the resolution of uncertainty is exogenous to the firm, the option-adjusted hurdle rate is greater than the traditional hurdle rate and increases with the level of uncertainty. In contrast, if the resolution of uncertainty is endogenous, the option-adjusted hurdle rate is lower than the traditional and decreases with the level of uncertainty. The existence of sunk costs, no matter how high, is not sufficient to cause the hurdle rate to be greater than traditional. Further, we show that embedded options have value even in the absence of sunk costs. The framework also allows a straightforward comparison between the traditional analysis and the option-adjusted analysis under uncertainty and the conditions under which the myopic traditional approach is true.

INFORMATION ARRIVAL AND REAL INVESTMENT

I Introduction

Recently, insights generated by option-theoretic work have been fruitfully applied to the evaluation of investment decisions under uncertainty. The main contribution of this literature has been in showing that the traditional NPV analysis can be misleading. The current literature can be divided into two main strands. The first strand, as discussed in the survey paper by Dixit (1992), has emphasized the importance of irrecoverable costs or the sunk costs of investment, showing that the correct hurdle rate should be greater than traditional and it should increase with uncertainty.[1] The second strand of literature, represented primarily by Roberts and Weitzman (1981) and Pindyck (1993), show that if the sunk costs were uncertain, it is possible that the optimal hurdle be less than the traditional. This could happen if the information arrival or the evolution of uncertainty is *endogenous*, in that it is affected by the action of the firm. In the former strand, it is *exogenous* in the sense that it cannot be controlled by the action of the firm.[2]

The literature thus suggests that there are *two* factors that are important in determining the nature of the hurdle rate: the nature of resolution of uncertainty and the level of sunk costs. Though the literature has tended to emphasize sunk costs (for example, see Bernanke (1983), McDonald and Siegel (1986) and Ingersoll and Ross (1992)), it would seem investment in new industries and technologies as well as mergers mergers are governed more by the nature of uncertainty than by inertia induced by the presence of sunk costs. It is therefore important to understand the relative importance of these two factors - what are the respective roles played by sunk costs and the evolution of uncertainty in determining the behavior of the hurdle rate and its relation with the traditional hurdle rate? The focus of this paper is to set up a framework that allows a transparent comparison between the sunk costs and the evolution of uncertainty, between the two types of uncertainty, and between the traditional analysis and the option-adjusted analysis under uncertainty.

We can summarize our results as follows. First, we show that the evolution of uncertainty is the sole factor that determines the behavior of the hurdle rate. Changing the uncertainty from exogenous to endogenous changes the hurdle rate from being greater than traditional to being less than traditional, ceteris paribus. This result also shows that the existence of sunk costs, *no matter how high*, is not sufficient to cause the hurdle rate to be greater than the traditional. The nature of uncertainty also determines whether the hurdle rate increases or decreases with the level of uncertainty.

Second, we show that sunk costs are not necessary for embedded options to have value. In all the papers with exogenous uncertainty, sunk costs are necessary for options to have value; the option to wait or abandon a project have no value in the absence of sunk costs (see amongst others, McDonald and Siegel (1985, 1986) and Dixit (1989)). This suggests that a more general way of looking at these embedded options is to consider them as a means of maintaining flexibility in the wake of new information.

Third, we clarify the role that sunk costs play in the investment decision. We can view the discrepancy between the option-adjusted hurdle rate and the traditional hurdle rate as arising from the failure of the traditional approach to explicitly account for the arrival of new information. Thus, the option-adjusted hurdle rate is a product of intertemporal optimization while the traditional hurdle rate is myopic. Sunk costs strengthen or weaken the intertemporal link between present and future action depending on the nature of uncertainty and thus the difference between the option-adjusted and the traditional hurdle rate. The level of sunk costs determines when the myopic traditional analysis is equivalent to that of intertemporal optimization. We show that when the uncertainty is endogenous, zero level of sunk costs make the traditional hurdle rate equivalent to the option-adjusted hurdle rate, as opposed to 100% if the uncertainty is exogenous.

The contribution of our model of endogenous uncertainty in comparison with the existing models of Roberts and Weitzmann (1981) and Pindyck (1993) is two-fold.[3]First, it allows a direct comparison with the existing generic model with exogenous uncertainty as in Dixit (1992). Second, it separates the effect of sunk costs from that of endogenous uncertainty and allows us to understand the specific role of these two factors. More generally, our paper adds to the literature that values flexibility and real options. The literature has valued options to wait and abandon (McDonald and Siegel (1985, 1986)), and other sources of managerial flexibility (Triantis and Hodder (1990)) and the interaction amongst these (Trigeorgis (1993)). Instead of focussing on any specific source of managerial flexibility, we emphasize the general importance of keeping flexibility in the wake of new information.[4]

The paper is organized as follows. In section 2, we define the firm's optimization problem, that sets the framework for the rest of the paper. We model the evolution of information in the simplest form; we assume that uncertainty can be represented by a single diffusion process. The 'learning' that occurs here is the change in the distribution of the revenue at each realization of the random variable. The implication is that, although the distribution of the revenue is currently known, it is expected that new information may require revision of beliefs regarding this distribution. The modelling choices are made such that it allows straightfoward comparison with the other papers in the literature.[5]

In sections 3 and 4, we characterize the solution of the firm's optimization problem under different circumstances. In section 3, the firm faces an investment project with endogenous evolution of uncertainty. For the purpose of contrasting with the results in section 3, we derive the results under an exogenous information arrival process in section 4. Section 6 concludes and offers simple examples of the application of the analysis.

II The Firm's Optimization Problem

We consider a risk-neutral firm that has the choice of investing in either of two projects. First, it may invest in a risky project, where the only source of risk is the uncertain nature of the cash flows of the investment project. Alternatively, the firm has the option of investing in a safe project with known cash flows, the return to which constitutes the opportunity cost or cost of capital of the firm. We normalize the size of the investment to 1. We assume that the firm has the option to postpone its investment, and that the quality of the investment project does not deteriorate if it is not taken up immediately.[6]The firm may also abandon the project at any time, recovering all but a fraction I of the original investment, $0 \leq I \leq 1$. This 'sunk' cost may be entry or set-up costs, or related to exit costs, perhaps due to an illiquid or imperfect market for asset liquidation or takeovers.

Let the revenue on the risky project, $R(p_t)$, be a function of the single state variable, p_t,where

$$dp_t = \mu p_t dt + \sigma p_t dz_t;\ p_0 \tag{1}$$

and z_t is standard Brownian motion that constitutes the only source of uncertainty in the investment project. As in Ross (1989), we interpret p_t as information that is perfectly correlated with the revenue of the project and let $R(p_t) = p_t$.[7]The firm's objective is to maximize the value of its investment, V, which is the discounted net present value of its cash flows, F_t, in expectation terms.

$$V = \max_{F_t} E \int_0^\infty e^{-rt} F_t dt \tag{2}$$

where $r > 0$ is the constant discount rate that is equivalent to the instantaneous return on the safe project, or, given risk-neutrality, equal to the riskless rate. The cash flows, F_t, may come from the risky or riskless project. The firm's optimization problem at any time t is to decide whether F_t should come from the revenue of the risky or the riskless project. Whenever the firm is not invested in the risky project, we assume that it earns the rate r in the riskless investment.

Consider the problem of the firm, (**??**), at t_0. The firm has to decide whether it should invest in the risky project, given the initial condition p_0 of the revenue process p_t. We define a hurdle rate, H, as that value of p_0 that makes the firm just indifferent between investing and not investing in the project. The optimal policy of the firm can now be analyzed in terms of this hurdle rate.[8]We shall show later that there exists a strict monotonic relationship between this hurdle rate and r. This allows us to define an equivalent hurdle rate on r (say, H'), such that, conditioned on p_0, the firm takes the project if r is less than H'.

We use this framework for the analysis of the optimal policy in sections 3 and 4. Thus, we ask the following question: Suppose all projects have sunk costs and can be postponed or abandoned, then what determines the behavior of the hurdle rate? To show that the information arrival process determines completely the nature of the hurdle rate, we examine each mode of information arrival seperately. In section 3, we model uncertainty as being completely endogenous to the firm, and, in section 4, as being exogenous to the firm. When the uncertainty is exogenous, the Brownian motion, z_t, evolves independently of the action of the firm. When the uncertainty is endogenous, z_t evolves only if the firm takes up the project. In the former case, z_t may be considered as the source of uncertainty that is truly independent to the firm, and may, for example, relate to macroeconomic variables. In the latter case, z_t may be construed as a source of uncertainty that is intimately tied to the firm and its investment project. It may pertain to any information that can be only be updated when the project is taken up. Examples of this could be uncertainty regarding the cost structures (Pindyck (1993)) or market reaction to the entry of the firm, synergies in the case of mergers, etc. We will subscript all variables in the endogenous case with N and, in the exogenous case, with X. We use the traditional analysis as a basis for comparison and use the subscript T for these variables.

Before we close this section, we derive the results of interest under the traditional analysis. Let V_T be the value of investing in the risky project. As the traditional analysis ignores the arrival of information, V_T is the worth of the project as if p_t evolved exactly as per the expectations at time t_0. This is simply $V_T = E\int_0^\infty e^{-rt} p_t dt = p_0/(r-\mu)$, where we require μ to be less than r for the integral to be finite, and we shall assume that to be so in all the subsequent analysis.

The traditional analysis requires that the firm take the project if $p_0/(r-\mu)$ is greater than the original investment. As we have normalized the investment to 1, the necessary condition for the firm to invest simplifies to $p_0 + \mu \geq r$. We define $H_T = r - \mu$ as the hurdle rate for the traditional analysis. Note that V_T and H_T are both independent of I and σ. This is not true in general, as we shall see below. We will use V_T and H_T as a basis for comparing the rest of the results in the rest of the sections.

Note that the traditional analysis does *not* ignore sunk costs or embedded options. It simply considers them irrelevant to the decision. In this context, we should stress that it is not simply the effect of uncertainty that constitutes the difference between the traditional analysis and the option-adjusted analysis (with either endogenous or exogenous uncertainty). For example, if returns were a random variable distributed with some known time-invariant distribution, then the traditional approach would still be true. It is only when the arrival of information results in changing beliefs regarding the distribution[9]that the traditional approach breaks down, as it ignores the impact of learning and new information. The inadequacy of the traditional approach stems from the fact that it is myopic and ignores the impact of new information on the current decision. However, there are circumstances when the myopic approach still yields correct results and we shall discuss this below.

III The Optimal Decision under Endogenous Information

In this section, we analyze the optimal policy of the firm when the evolution of uncertainty is endogenous. We set up the model and find an analytic solution that allows us to derive the main results of this paper. We also consider some numerical results.

A. The Model and its Analytic Solution

When uncertainty evolves endogenously, the firm can only update its information if it invests. Suppose t' is the last time the firm invested in the project. In that case, the information that the firm has at any time $t > t'$ is the same as its information at time t'. In particular, if the firm has not invested in the project at any $t' > 0$, then its information at time t is the same as its information at t_0, which corresponds to the initial condition, p_0.[10]

We can use this reasoning to simplify the decision problem of the firm by noting that if it is not optimal for the firm to invest at t_0, it would not be optimal at any time, $t > t_0$. The problem of the firm is then simply to determine (1) whether or not it should invest at t_0,[11] and (2) conditioned on investing at t_0, when it should exit. Consider the worth of the project, $V_N(p_t; I)$, conditioned on the firm investing at t. The optimal decision now only involves determining the time, τ, when it is optimal for the firm to exit out of the risky project and take up the riskless project. The time τ constitutes a non-anticipatory stopping time for the project, where given the informational assumptions, stopping occurs only once. The payoff upon stopping is thus known and is equal to $(1-I)e^{-r\tau}$. Consequently, the optimization problem for the firm, (**??**), may be represented as a simple stopping time problem for determining $V_N(p_t; I)$.

$$V_N(p_t; I) = \sup_{\tau} E \int_t^{\tau} e^{-r(s-t)} p_s ds + (1-I)e^{-r(\tau - t)} \tag{3}$$

The firm will invest at t_0 if $V_N(p_0; I) > 1$. We will sometimes suppress the functional dependence of $V_N(p_t; I)$ for readability. We use the subscript N for all the variables that are specific to the information being endogenous.

Consider the solution to the firm's problem as in (**??**). We convert the stopping time problem into a free boundary problem in the usual way.[12] The parabolic partial differential equation reduces to an ordinary differential equation, as t only enters in the discount factor. We look for a number $P(I)$[13] such that the continuation region will be the interval (P, ∞) and a function $V_N(p; I)$ that satisfies:

$$\begin{aligned} \frac{1}{2}\sigma^2 p^2 V_N''(p) + \mu p V_N'(p) - r V_N(p) + p &= 0; p \geq P \\ &\leq 0; p < P \end{aligned} \tag{4}$$

with the following boundary conditions,

$$V_N(P) = 1 - I; \tag{5}$$

$$V_N'(P) = 0; \tag{6}$$

$$V_N(p) < \infty, \forall p \tag{7}$$

$$V_N(p) > 1 - I; p > P \tag{8}$$

In the appendix, we solve for $V_N(p; I)$ as the solution to the second-order ordinary differential equation, subject to the boundary conditions. We need three boundary conditions to identify the solution. The first, (**??**), is the value of the investment when the firm abandons the project. The second, (**??**), is the usual smooth-pasting or 'high- contact' boundary condition that along with (**??**) uniquely identifies P.[14]The final boundary condition, (**??**), determines the limiting behavior of V_N as $p \to \infty$.[15]We need to verify the final solution, using (**??**), to ensure that the solution is reasonable.

The solution consists of two parts, the solution to the homogeneous part of the differential equation combined with a particular solution. We show the following in the appendix:

Lemma 1 *The function that solves (**??**) is:*

$$V_N(p) = \begin{cases} V_T - \frac{1}{(r-\mu)\lambda_1} P^{1-\lambda_1} p^{\lambda_1} & \text{if } p > P \\ 1 - I & \text{otherwise} \end{cases} \tag{9}$$

where

$$P = \frac{(\mu - r)(1 - I)\lambda_1}{1 - \lambda_1} \tag{10}$$

$$\lambda_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} \tag{11}$$

Further, V_N is a decreasing function of I. Furthermore, $V_N = V_T$ if $I = 1$, and for I strictly less than 1, V_N is strictly greater than V_T.

V_N may thus be viewed as consisting of V_T, plus a positive (as λ_1 is negative) premium. Recall that V_T is equivalent to the worth of the project if p_t evolved exactly as per time t_0 expectations.[16]The premium, therefore, may be viewed as the additional value that results from the arrival of new information. Though the magnitude of the premium is related to the gains from information, the latter is *not* sufficient to guarantee a premium. As in all real option models, the reason why the premium exists is because of irreversibility or loss of flexibility. The firm values the need to keep flexibility in the wake of new information. By not investing, access to new information is cut off and therefore the decision *not* to invest is irreversible. This may be contrasted with the existing models with exogenous information where a decision to invest is irreversible if the firm cannot recover its original investment completely. The important point to note is that sunk costs are not required to induce irreversibility. Further, the investment decision of the firm is more complex than thought as both the decision to invest or not invest can be costly and irreversible.

As the source of flexibility in our case is the ability to abandon the project, we may also view the premium as an option to abandon. However, this nomenclature has the potential to be misleading for two reasons. First, we could have let the firm have some other source of managerial flexibility and still derived similar results. Second, simply being able to abandon the project does not make the option valuable. We have already discussed that if the distribution of the stochastic returns was stable and known, as in the traditional analysis, the option to abandon is not valuable. Another example is the case when uncertainty is exogenous. As Dixit (1989) clarifies, when uncertainty is exogenous, the option to abandon is not valuable when sunk costs are zero, which may be contrasted with our model by letting $I = 0$. In every case, the option to abandon (as well as any other source of managerial flexibility), has value only when the advent of possible new information creates a need to keep flexibility.

One may also nest the traditional analysis within this model by examining conditions when the myopic analysis is correct. The traditional analysis will be equivalent to the option-adjusted analysis if new information has no effect on the firm's actions. This may occur for two reasons. First, by simply making the project irreversible, we remove all flexibility of action. In this case, new information has no value as the firm cannot act on it and this reduces the intertemporal optimization problem to the myopic one. Alternatively, by letting $p_0 \to \infty$, we again reduce the analysis to the traditional one. In this case, since the project is ex ante very profitable, new information does not have much value. The importance of this analysis is for projects that are marginal, as the possibility of receiving information provides an incentive for the firm to invest in projects that it would not otherwise have.

Though we have assumed the initial condition p_0 to be known by the firm, we may relax this assumption by instead allowing the firm to know the distribution of p_0. Given that P (and thus the optimal stopping time) is independent of p_0, the same P remains optimal as the initial condition is randomized. Thus the worth of the project when p_0 is stochastic can simply be obtained by integrating V_N from the lemma over the distribution of p_0. Therefore, in all our results below, one may replace p_0 by its mean.

Clearly, the firm will take the investment project if $V_N(p_0)$ is greater than the original investment. This gives us the criteria that the firm may use as an optimal policy. We define a hurdle rate H_N as that value of $\hat{p}$ such that $V_N(\hat{p}) = 1$. The firm will take the project if $p_0 \geq H_N$. We show the following in the appendix:

Proposition 1 *There exists a unique hurdle rate, H_N. If $I = 0$, $H_N = P$. If $I = 1$, $H_N = H_T$. For I strictly less than unity, H_N is strictly less than H_T.*

The results stated in the proposition follow almost immediately from the analytic solution for V_N. It is, however, worth emphasizing two points. First, in all cases when $I < 1$, the hurdle rate H_N is *less* than H_T, a result which we shall contrast with the hurdle rate derived under exogenous information in the next section. It is this conclusion that shows how the information arrival process is crucial in determining the investment criteria of the firm. Second, it follows from the result that that although sunk costs, I, are important in determining the exact hurdle rate, they do not determine whether the correct hurdle rate should be greater or less than the traditional. This, combined with the earlier observation that sunk costs are not the only source of irreversibility, suggest that the role of sunk costs in the capital budgeting decision is contributory rather than pivotal.

The next proposition gives the comparative statics of H_N and V_N:

Proposition 2 $\partial H_N/\partial\sigma < 0$, $\partial H_N/\partial I > 0$, $\partial H_N/\partial r > 0$.

Also, $\partial V_N/\partial\sigma > 0$, $\partial V_N/\partial I < 0$, $\partial V_N/\partial r < 0$..

Increase in uncertainty has an unambiguously positive effect on the value of the firm. It increases the worth of the project the firm has, by increasing the gains from learning. This adds a new layer of complexity to the capital budgeting decision by setting up a tradeoff between a project with a high mean and low variance and another with a low mean and high variance. In a sense, this is the intertemporal tradeoff between current payoff and possible future gains.

The main implication of this proposition is that increase in uncertainty increases the propensity to invest, a conclusion that may be contrasted with that of the 'waiting to invest' literature with exogenous uncertainty. This implication also highlights the potential difficulty in establishing a clear empirical relationship between investment and uncertainty. For example, foreign direct investment may be driven as much by the desire to learn more about the foreign country (endogenous uncertainty) as by the fluctuations in the exchange rate (exogenous uncertainty). This suggests that any empirical tests of real option models should be carefully chosen on industries which do not have the confounding effects of both types of uncertainty.

The other comparative statics results reported in the proposition are straightforward and do not require disussion. To summarize, we have shown that when information is endogenous the hurdle rate is less than the traditional and decreases as uncertainty increases. We next consider some numerical results.

B. Numerical Results

We present numerical results for certain parameter values. In all our calculations we assume the interest rate, r to be 5%, and $\mu = 0$.[17] Thus, $H_T = 0.05$. In Table 1, we tabulate values for the hurdle rate, which may be interpreted in two different ways. First, it may be interpreted as defined earlier in the section, so that, conditioned on r, the hurdle rate is defined as H_N which is the value of p_0 that makes the firm indifferent between investing and not investing in the project. We can also define an equivalent measure, H_T', conditioned on $p_0 = H_N$, such that the firm takes the project if r is less than or equal H_T', under the traditional analysis. Thus, we define $H_T' = H_N + \mu$.

To take an example, suppose H_N is 0.03, so that the firm takes the project if p_0 is greater than or equal to H_N. In contrast, under the traditional analysis, the firm would take the project if $p_0 \geq 0.05$ or, conditioned on $p_0 = 0.05$, if $r \leq 0.05$. Now suppose $p_0 = H_N = 0.03$. Then if the firm evaluated the project by the traditional analysis conditioned on p_0, it should correctly take the project if $r \leq 0.03$. Thus, we define H_T' as equal to 0.03, which may be compared with the earlier wrong cutoff of $H_T = 0.05$, the implication being that the firm should use a cost of capital of 3% instead of 5%. The advantage of normalizing μ to zero is that it allows us to easily interpret the hurdle rate as either H_N or H_T'.

In Table 1, we tabulate the hurdle rate for a range of values of σ and I. Typical estimates of the volatility of price of the equity of the company would be between 20% to 40%, and one may plausibly expect the revenue to be at least that volatile. For new industries and technologies, one may expect volatilies of several times the magnitude of more mature industries. For $\sigma = 0.2$ and sunk costs of about 10%, the correct hurdle rate is 25% lower than the traditional. For $\sigma = 0.6$, it is 60% lower, so that if the firm analyzes the project using traditional methods, it should consider $r = 2\%$, instead of 5%. At this level of volatility, even with sunk costs equal to 50%, the true hurdle rate is 25% lower.

Consider next the limiting behavior of the hurdle rate. The limiting behavior as $I \to 1$ has already been discussed; the hurdle rate reduces to the traditional. We can also consider analytically the limiting behavior of V_N and H_N as $\sigma \to \infty$. Noting that $\lambda_1 \to 0$ as $\sigma \to \infty$, we get

$$\begin{aligned} \lim_{\sigma \to \infty} V_N &= \lim_{\lambda_1 \to 0} V_T - \frac{1}{(r-\mu)\lambda_1} P^{1-\lambda_1} p^{\lambda_1} \\ &= V_T + 1 - I \end{aligned}$$

where we use (11) and the fact that $\lim_{\lambda_1 \to 0} \lambda_1^{-\lambda_1} = 1$. The hurdle rate, H_N, is thus equal to

$(r - \mu)I$. Thus, instead of comparing the worth of the project with the original investment as in the traditional analysis, the firm only needs to compare with the level of irrecoverable costs. This behavior seems to be indicative of actual behavior in new industries like genetic engineering, where the potential amount that investors can lose seems to be an important decision variable in evaluating the investment decision. An interesting implication that follows is that if the project has zero sunk costs then, irrespective of the cost of capital r, it is *always* optimal to take the project.

In the next section, we shall analyze the firm's optimal policy in the exogenous case.

IV The Optimal Decision under Exogenous Information

In this section, we assume that the uncertainty is exogenous. As noted, this case has been extensively explored in the literature. We, therefore, limit the analysis to that required for comparison with the previous section.

The analysis here is more complex than the preceding as the optimal decision involves more than one stopping time. First, since the uncertainty evolves without the firm having to take any action, there is value to waiting for more information to arrive. The first stopping time problem is thus to determine the optimal time to stop waiting and invest. Once the firm has invested, as in the preceding section, it has the option of disinvesting if it gets enough unfavorable information and we have the stopping time problem similar to that of the previous section. However, in the previous section, once the firm disinvests, it never invests again. Here the whole cycle of investing and abandoning can repeat infinitely often.

Let $V_{X,0}$ be the worth of the firm when it is not invested in the risky project and $V_{X,1}$ be the worth of the firm when it is invested. Let τ_0 be the optimal time to invest in the project and τ_1 be the optimal stopping time to exit the project. Then the two stopping time problems that constitute the optimization problem of the firm are:

$$V_{X,0}(p_t; I) \quad = \quad \sup_{\tau_0} E e^{-r(\tau_0 - t)} V_{X,1} - I \tag{12}$$

$$V_{X,1}(p_t;I) = \sup_{\tau_1} E\int_t^{\tau_1} e^{-r(s-t)}p_s ds + e^{-r(\tau_1-t)}V_{X,0} \tag{13}$$

where we assume that the sunk cost I is incurred at entry. The firm will invest only if $V_{X,1}$ is greater than $V_{X,0}$ and therefore the hurdle rate, H_X is that value of p at which $V_{X,0}(p) = V_{X,1}(p)$. The two stopping time problems have to be solved simultaneously, after converting them into equivalent free boundary problems. It is not possible to get a closed form solution to this model.[18]Instead, we will make the simplifying assumption that, if the firm exits, it will never re-enter the project. This assumption is not critical and allows us to derive analytical results. In this case, the optimization problem of the firm reduces to:

$$V_{X,1}(p_t;I) = \sup_{\tau_0} Ee^{-r(\tau_0-t)}V_T \tag{14}$$

The firm will invest if $V_{X,1} > V_{X,0}$ and this determines the hurdle rate. The equivalent free boundary problem is :

$$\begin{aligned}\frac{1}{2}\sigma^2 p^2 V''_{X,0}(p) + \mu p V'_{X,0}(p) - rV_{X,0}(p) &= 0; p < Q \\ &< 0; p > Q \end{aligned} \tag{15}$$

where Q is the value of p at which the firm is indifferent between $V_{X,0}$ and $V_{X,1}$ and is, therefore, the hurdle rate, H_X. The boundary conditions are:

$$V_{X,0}(Q) = V_{X,1}(Q) - I; \tag{16}$$

$$V'_{X,0}(Q) = V'_{X,1}(Q); \tag{17}$$

$$V_{X,0}(p) > V_{X,1}(p) - I; p < Q \tag{18}$$

$$V_{X,0}(p) < \infty, \forall p \tag{19}$$

$$\tag{20}$$

Consider the case when $I = 1$. It is straightforward to solve for $V_{X,0}$.[19]

$$V_{X,0}(p) = \begin{cases} \frac{1}{r-\mu} Q^{\lambda_2 - 1} p^{\lambda_2} & \text{if } p < Q \\ V_T & \text{otherwise} \end{cases} \tag{21}$$

where

$$Q = H_T \frac{\lambda_2}{\lambda_2 - 1} \tag{22}$$

$$\lambda_2 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} \tag{23}$$

It follows that Q -the hurdle rate, H_X- is strictly greater than H_T and H_N (as λ_2 is greater than 1). Further, since λ_2 decreases with σ, H_X increases with uncertainty.

It is more of interest to the examine the behavior of the hurdle rate as a function of I. Though we cannot analytically solve for Q, we can analytically examine its relationship with I. When $I = 0$, clearly there is no value to waiting as the firm can always act on future information, irrespective of its current decision. The problem then reduces to the static problem and the relevant hurdle rate reduces to that of the traditional analysis, $H_X = H_T$. We can use this observation to show that H_X is strictly greater than H_T for all $I > 0$ by proving in the appendix that Q is a monotonically increasing function of I. Therefore, the lowest hurdle rate is at $I = 0$, when $H_X = H_T$ and for all $I > 0$, $H_X > H_T$. This result can be contrasted with that of proposition 1, where we had proven that the hurdle rate H_N was *less* than H_T. We summarize this in the following proposition:

Proposition 3 $H_N \leq H_T \leq H_X$

Proof: See Appendix.

This proposition summarizes the main results of the paper. The resolution of uncertainty is instrumental in determining whether the option-adjusted hurdle rate is greater or less than the traditional analysis as well as whether it increases or decreases with uncertainty. The role of sunk costs is then to determine the magnitude of the discrepancy between the traditional and option-adjusted hurdle rate as well as the circumstances when the myopic traditional approach is true.

V Conclusion

We have constructed a simple, generic model that allows a transparent comparison between the two modes of uncertainty, between the respective roles played by uncertainty and sunk costs as well as between the traditional analysis and the option-adjusted analysis under uncertainty. Our model shows that the behavior of the hurdle rate is defined by the way uncertainty resolves. If uncertainty is exogenous, the hurdle rate is greater than the traditional and increases with uncertainty, while if it is endogenous, it is less than traditional and decreases with uncertainty. These conclusions hold true irrespective of the level of sunk costs. Further, we showed that embedded options may have value even in the absence of sunk costs. We also characterized the circumstances when the myopic traditional analysis is equivalent to that of intertemporal optimization. Specifically, we show that high sunk costs cause a small discrepancy for endogenous uncertainty as compared with large discrepancy between the traditional and option-adjusted hurdle rate for exogenous uncertainty and In the limits, the myopic approach is equivalent to the analysis under uncertainty if sunk costs are 100% and uncertainty is endogenous, and if sunk costs are 0% and the uncertainty is exogenous.

An empirical implication of the model is that industry dynamics would be very different in industries characterized largely by endogenous uncertainty (like genetic engineering or multimedia) as opposed to those characterized by exogenous uncertainty. One testable proposition is that industries with exogenous uncertainty would be characterized by inertia, while industries characterized by endogenous uncertainty would be characterized by firms rapidly entering and exiting the industry. The average time between entry and exit would be high for a firm in an industry with exogenous uncertainty and low for a firm in an industry with endogenous uncertainty. It would also be interesting to investigate the implications of the analysis in understanding the role of venture capitalists and the kind of financial contracts that are optimal when information is either endogenous or exogenous.

One may also apply these results in more unconventional settings. For example, Dixit (1989) cites the reluctance of universities to approve new faculty positions in departments that experience a surge in students as an application of the 'option to wait', arising because of exogenous uncertainty. We can use the same example as an application for endogenous uncertainty: Given that the university is hiring, note the enthusiasm to hire new faculty straight out of the Ph.D. program, and the subsequent high rate of tenure refusals. If the exogenous uncertainty regarding the future student population makes universities reluctant to hire new faculty, once the decision to hire has been made, the endogenous uncertainty regarding the capabilities of the young Ph.D. makes the university over-invest in them.

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Appendix

Proof of Lemma 1:

We solve the control problem by converting the optimal stopping time problem, (**??**), into the free boundary problem, (**??**).

We look for a number P(I) and a bounded, function $V_N(p; I)$ such that the continuation region will be the interval (P, ∞) and $V_N(\cdot)$ satisfies:

$$\begin{aligned}\frac{1}{2}\sigma^2 p^2 V_N''(p) + \mu p V_N'(p) - rV_N(p) &= -p; p > P \\ &< -p; p < P \qquad (24)\end{aligned}$$

where $\mu \leq r$ for V_N to be bounded.

$$\begin{aligned}V_N(P) &= 1 - I; \\ V_N'(P) &= 0; \\ V_N(p) &> 1 - I; p > P \\ V_N(p) &< \infty, \forall p\end{aligned}$$

The solution of this second-order differential equation will consist of a solution to the homogeneous part, combined with a particular solution.

Consider first the homogeneous part of **??**:

$$\frac{1}{2}\sigma^2 p^2 V''(p) + \mu p V_N'(p) - rV_N(p) = 0 \qquad (25)$$

To solve **??**, consider p^λ as a possible solution and the corresponding characteristic equation:

$$\frac{1}{2}\sigma^2 \lambda(\lambda - 1) + \mu\lambda - r = 0 \qquad (26)$$

Let the roots of the characteristic equation be λ_1 and λ_2,

$$\lambda_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}$$
$$\lambda_2 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} \tag{27}$$

Note that both roots are real and, further, λ_1 is negative and λ_2 is greater than one. The last conclusion follows as we have assumed $\mu < r$.

The general solution of the homogeneous equation, $V(p)$, is then,

$$\begin{aligned} V(p) &= c_1 V_1(p) + c_2 V_2(p) \\ &= c_1 p^{\lambda_1} + c_2 p^{\lambda_2} \end{aligned} \tag{28}$$

where c_1 and c_2 are constants to be determined. Note that V_1 and V_2 are linearly independent solutions of V_N as the Wronskian, W:

$$\begin{aligned} W &= \begin{vmatrix} V_1 & V_1' \\ V_2 & V_2' \end{vmatrix} \\ &= (\lambda_2 - \lambda_1) p^{(\lambda_2 - \lambda_1) - 1} \\ &\neq 0, as \lambda_1 \neq \lambda_2 \end{aligned} \tag{29}$$

It may be verified that **??** is indeed a solution to the homogeneous equation for any constants c_1 and c_2.

To obtain the general solution to the non-homogeneous equation, we need to add a particular solution, V_0 of **??** to **??**. Using standard procedure to find the particular solution, hypothesize $V_0 = \alpha_1(p)V_1(p) + \alpha_2(p)V_2(p)$ and $\alpha_1'(p)V_1(p) + \alpha_2'(p)V_2(p) = 0$. Next, noting that V_0 satisfies **??**, we get:

$$\alpha_1' V_1' + \alpha_2' V_2' = -p \tag{30}$$

Combining this equation with the earlier $\alpha_1' V_1 + \alpha_2' V_2 = 0$ enables us to solve explicitly for $\alpha_1(p)$ and $\alpha_2(p)$:

$$\begin{aligned}
\alpha_1(p) &= \int \frac{pV_2}{W} dp \\
\alpha_2(p) &= \int \frac{-pV_1}{W} dp
\end{aligned}$$

where W is the Wronskian seen earlier in (??). Solving for α_1 and α_2, we get the particular solution as:

$$V_0(p) = \frac{p}{r - \mu} \tag{31}$$

The solution to (??) is the general solution to the homogeneous equation combined with the particular solution:

$$\begin{aligned}
V_N(p) &= V_0(p) + c_1 V_1(p) + c_2 V_2(p) \\
&= \frac{p}{r - \mu} + c_1 V_1(p) + c_2 V_2(p)
\end{aligned} \tag{32}$$

We need to use the boundary conditions to solve for c_1 and c_2. First, note that since $V_N(p)$ is bounded, the coefficient c_2 of V_2 must be 0, as V_2 is the solution that corresponds to the positive root, $\lambda_2 > 1$, of the characteristic equation. Next, using the smooth-pasting condition and the boundary condition corresponding to the value of V_N at absorption, we can solve for the two remaining unknowns, c_1 and P:

$$\begin{aligned}
V_N(P) &= 1 - I = \frac{P}{r - \mu} + c_1 V_1(P) \\
V_N'(P) &= 0 = \frac{1}{r - \mu} + c_1 V_1'(P)
\end{aligned} \tag{33}$$

Thus, we get:

$$P = \frac{(\mu - r)(1 - I)\lambda_1}{1 - \lambda_1} \tag{34}$$

$$c_1 = \frac{1}{(\mu - r)\lambda_1} P^{1-\lambda_1} \tag{35}$$

The final solution to **??** is:

$$V_N(p) = \begin{cases} \frac{p}{r-\mu} - \frac{1}{(r-\mu)\lambda_1} P^{1-\lambda_1} p^{\lambda_1} & \text{if } p > P \\ 1 - I & \text{otherwise} \end{cases} \tag{36}$$

where P is given by (**??**). Noting that the first term in V_N (corresponding to the particular solution) is also equal to V_T proves the first part of the proposition. The second part follows since, from (**??**), P is a strictly decreasing function of I and, therefore, so is V_N. Further, $P = 0$ when $I = 1$ and it follows immediately that $V_N = V_T$. Finally, as V_T is independent of I, it follows that V_N is strictly greater than V_T for all $I < 1$.

Proof of Proposition 1:

We will first prove existence. We need to show that there exists a $\hat{p}$ such that $V_N(\hat{p}) = 1$. From proposition 1, $V_N(p) = 1 - I$, for $p < P$. Also, $V_N(p) \geq 1$, for $p = r - \mu$. As V_N is continuous, it follows from the intermediate value theorem that there exists a $\hat{p}$ such that $V_N(\hat{p}) = 1$. By definition, $H_N = \hat{p}$.

To show uniqueness, note that V_N is strictly increasing for $p > P$, as:

$$\begin{aligned} V_N'(p) &= \frac{1}{r-\mu} - \frac{1}{r-\mu}\left(\frac{P}{p}\right)^{1-\lambda_1};\ p > P \\ &> 0 \end{aligned} \tag{37}$$

As V_N is strictly monotonic, it follows that V_N takes the value 1 only once, and this proves uniqueness of H_N.

We next prove the remaining statements of the proposition. First, consider the case of $I = 0$. In this case, from proposition 1 the value $V_N(P) = 1$ and it follows that the hurdle rate is P. Next, consider $I = 1$. In this case, $P = 0$, and $V_N = V_T$ and thus $H_N = H_T$. Finally, as H_N is that value of $p > P$ at which $V_N = 1$, and as P (and, therefore, V_N) is a decreasing function of I, it follows that for all $I < 1$, $H_N < H_T$. □

Proof of Proposition 2

We need to calculate $\frac{dP}{d\sigma^2}$ and $\frac{d\lambda_1}{d\sigma^2}$:

$$\begin{aligned}\frac{d\lambda_1}{d\sigma^2} &= \frac{\mu}{\sigma^4} - \frac{1}{2\sqrt{(\mu-\frac{1}{2}\sigma^2)^2+2r\sigma^2}}\left(\frac{-2\mu^2}{\sigma^6} - \frac{2r-\mu}{\sigma^6}\right) \\ &> 0 \end{aligned} \tag{38}$$

where λ_1 is given in (**??**). Finally, we calculate $\frac{dP}{d\sigma^2}$:

$$\begin{aligned}\frac{dP}{d\sigma^2} &= (\mu-r)(1-I)\frac{d}{d\sigma^2}\frac{\lambda_1}{1-\lambda_1} \\ &= (\mu-r)(1-I)\frac{\frac{d\lambda_1}{d\sigma^2}}{(1-\lambda_1)^2} \\ &= P\frac{1}{\lambda_1(1-\lambda_1)}\frac{d\lambda_1}{d\sigma^2}\end{aligned} \tag{39}$$

Now,

$$\frac{\partial V_N}{\partial\sigma^2} = \frac{\partial V}{\partial P}\frac{\mathrm{d}P}{\mathrm{d}\sigma^2} + \frac{\partial V}{\partial\lambda_1}\frac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma^2} \tag{40}$$

Substituting from (40) and (41) into (42), we get

$$\begin{aligned}\frac{\partial V_N}{\partial\sigma^2} &= -\frac{1}{(r-\mu)\lambda_1}P^{1-\lambda_1}p^{\lambda_1}\log\frac{p}{P}\frac{d\lambda_1}{d\sigma^2} \\ &> 0\end{aligned}$$

It follows that $\frac{\partial H_N}{\partial\sigma^2}$ is negative.

The comparative statics results with respect to I and r follow in the same fashion as $\frac{\partial P}{\partial I}$ is negative and $\frac{\partial P}{\partial r}$ is positive. □

Proof of Proposition 3

$V_{X,0}$ satisfies the following equations:

$$\frac{1}{2}\sigma^2 p^2 V_X''(p) + \mu p V_X'(p) - rV_X(p) \begin{array}{ll} = & 0; p < Q \\ < & 0; p > Q \end{array} \tag{41}$$

where Q is the value of p at which the firm is indifferent between $V_{X,0}$ and $V_{X,1}$.

$$\begin{aligned} V_{X,0}(Q) &= V_{X,1}(P) - I; \\ V'_{X,0}(Q) &= V'_{X,1}(Q); \\ V_{X,0}(p) &> V_{X,1}(p) - I; p < Q \\ V_{X,0}(p) &< \infty, \forall p \end{aligned} \tag{42}$$

The solution to (41) is the same as that of the homogeneous part of (**??**). Thus,

$$V_{X,0}(p) = c_3 p^{\lambda_1} + c_4 p^{\lambda_2} \tag{43}$$

where λ_1 and λ_2 are defined as in (27) and c_3 and c_4 are constants to be determined. As V_X is finite, it follows that $c_3 = 0$. We thus need to determine c_4 and Q from the smooth-pasting condition and the value of $V_{X,0}$ at absorption. Further, the functional form of $V_{X,1}$ is the same as that of V_N:

$$V_{X,1} = \frac{p}{r-\mu} + c_5 p^{\lambda_2} \tag{44}$$

As $V_{X,1}$ and $V_{X,0}$ are both monotonically increasing in p, the constants c_4 and c_5 are positive. Combining (42),(43) and (44), we get:

$$\begin{aligned} c_4 Q^{\lambda_2} &= \frac{Q}{r-\mu} - c_5 Q^{\lambda_1} - I \\ c_4 \lambda_2 Q^{\lambda_2 - 1} &= \frac{1}{r-\mu} - c_5 \lambda_1 Q^{\lambda_1 - 1} \end{aligned} \tag{45}$$

Therefore, Q is the solution of the following equation:

$$\frac{Q}{r-\mu}(1-\frac{1}{\lambda_2}) = c_5 Q^{\lambda_1}(1-\frac{\lambda_1}{\lambda_2}) + I \tag{46}$$

Differentiating with respect to I, we get

$$\frac{dQ}{dI}[\frac{1}{r-\mu}(1-\frac{1}{\lambda_2}) - c_5 \lambda_1 Q^{\lambda_1 - 1}(1-\frac{\lambda_1}{\lambda_2})] = 1 \tag{47}$$

As the term within the square brackets is positive, it follows that $dQ/dI > 0$. As $Q = H_T$ at $I = 0$, it follows that for $I > 0$, $H_X > H_T$, where we designate Q as H_X. Combining this result with proposition 2, it follows that $H_N \leq H_T \leq H_X$. □

TABLE I

Hurdle Rates Under Endogenous Uncertainty

Hurdle rate, H_N or H_T'. The following parameter values are used: $r = 0.05$ and $\mu = 0$. The hurdle rate is shown for a range of values of the level of sunk costs, I, and volatility, σ.

	$I = 0$	$I = 0.05$	$I = 0.10$	$I = 0.5$	$I = 1.0$
$\sigma = 0.2$	0.0269	0.0345	0.0375	0.0473	0.050
$\sigma = 0.4$	0.0152	0.0230	0.0265	0.0417	0.050
$\sigma = 0.6$	0.0092	0.0164	0.0199	0.0373	0.050
$\sigma = \infty$	0	0.0025	0.0050	0.0250	0.050

NOTES

1. The policy implications arise from the firm having an "option to wait".

2. For example, in Ingersoll and Ross (1992), the exogenous variable is the stochastic interest rate while in in Dixit (1992) and Pindyck (1991), the value of the project or the price of the output is assumed to be exogenous. Other papers where the stochastic variable is exogenous are Cukierman (1980), Bernanke (1983), Brennan and Schwartz (1985), and McDonald and Siegel (1986).

3. Other papers that are in the same genre are Grossman, Kihlstrom and Mirman (1977) and Zeira (1987). These papers, including ours, fall in the more general category of problems known as bandit problems. For an introduction to the literature, see Berry and Fristedt (1985) and Gittins (1989) and, for a continous time formulation, see Karatzas (1984).

4. The general notion that flexibility adds value may break down when there are game-theoretic issues involved. See Kulatilaka and Marks (1988) for an example.

5. See Dixit (1992) and Pindyck (1991).

6. This assumption is required to make the problem non-trivial when the uncertainty is exogenous and is a standard assumption in the literature. It is not required for the analysis in section 3, when the uncertainty is endogenous.

7. Alternatively, we can allow a physical interpretation of p_t as the price in a competitive market of the output of the project. Normalizing the output of the project to a constant unity , it follows that $R(p_t) = p_t$.

8. Note that defining the hurdle rate in this way is equivalent to defining a hurdle rate on the mean of the distribution of p_t (and, therefore, R_t) as $E(p_t/p_0) = p_0 e^{\mu t}$.

9. Note that in the diffusion models, the conditional distribution at $t + T$ depends on p_t and thus potentially changes every period. In a discrete time model, one could achieve similar results by having a Bayesian updating of beliefs, as in Cukierman (1980).

10. We have assumed that the initial condition is known. This condition may be relaxed, as we shall see later.

11. In general, this is not true. For example, when the information is exogenous, the firm may enter and exit the project several times.

12. See, for example, Malliaris and Brock (1982), theorem 13.1, pp 127 which follows Van Moerbeke (1976), theorem 1. The theorem states that $\partial V_N/\partial t = 1/2\sigma^2 p^2 V_N'' + \mu p V_N'$. This can be simplified using (**??**).

13. The correspondence between P and the stopping time τ of (**??**) is as follows:

$$\tau = \inf(t \geq 0; V_N(p_t = P) = 1 - I) \tag{48}$$

14. For example, see Merton (1973). The time- independent nature of the boundary conditions makes our problem relatively easy to solve as opposed to the free boundary problem of the American option of finite maturity.

15. This condition is important for two reasons. First, if V_N was unbounded, then a unique solution may not exist. Second, economic considerations require that V_N converge to V_T as $p_0 \to \infty$ and $\tau \to \infty$.

16. In our problem, V_T is the particular solution of the differential equation, as might have been expected since a solution to the firm's optimization probem is $\tau = \infty$ for which $V_N = V_T$.

17. These values are similar to those considered in Dixit (1992).

18. Dixit (1989, 1991) solves it numerically for certain parametrizations and, also, provides some analytical approximations.

19. See Dixit (1992).